Frequency-Dependent Relaxation Time of Electron-Phonon Systems*

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(Received 22 April 1963)

A derivation of the frequency-dependent relaxation time in a simple model of metal, composed of electrons and phonons, is presented. The problem is investigated by a quantum-kinetic description of the response of the electron-phonon system to an oscillating electric field. The treatment, which stems from a proper time-dependent transport equation, does not have the time scale restriction of the usual transition probability approach, and does give a proper description of the time-dependent collective effects.

I. INTRODUCTION

IN the present paper we are concerned with the problem of calculating the frequency-dependent conductivity and relaxation time of a simplified model of a metal, composed of electrons and phonons, on the basis of a kinetic approach. The problem of electrical conductivity is usually treated using transport equations for the distribution function, f, under the provision that the rate of change of f due to the acceleration by a *time-independent* uniform electric field is balanced by the rate of change due to collisions.[1-3] Adopting from the classical theory of gases, the conductivity σ is given by

$$\sigma_d = ne^2\tau/m, \tag{1}$$

in terms of a characteristic *relaxation time*, τ, which is calculated, in one way or another, using the transport equation. In Eq. (1) m and $-e$ represent, respectively, the mass and charge of an electron, and n is the average density of the electrons. In the case of *time-dependent* fields, oscillating with the frequency ω, it is argued, after Drude,[4] that one can replace Eq. (1) by

$$\sigma(\omega) = \frac{ie^2 n}{m(\omega + i/\tau)}, \tag{2}$$

where, now, τ is generally frequency-dependent. Equation (2) is often introduced in discussing the interaction of electromagnetic waves with metals, especially in the infrared region, where $\omega\tau$ is greater than, or of order of, one. In order to find $\tau(\omega)$, one is not entitled to employ the *static* transport equation mentioned above, but has to search for a time-dependent method. The use of the *conventional* transport theory involves Dirac's time-dependent perturbation method, and the concept of *transition probability* is introduced under the assumption

that the collision frequency is the largest one under consideration. It is clear that this assumption does not hold for the present problem.

Some attention was paid to this problem in the last decade. Holstein[5] calculated the volume absorption of electromagnetic waves (in the infrared region) by employing a time-dependent perturbation theory. Ginzburg and Silin[6] considered the frequency-dependent conductivity, but assumed τ to be independent of frequency. Gurzhi[7] studied the proper transport equation to be used for the problem, but failed to obtain $\tau(\omega)$. In a recent series of papers, Ehrenreich[8] and his collaborators studied the optical properties of solids using (for the few eV range) an equation like (2) without specifying $\tau(\omega)$.

Here we shall investigate the frequency-dependent relaxation time, starting from the Hamiltonian of Bardeen and Pines[9] for a simplified model of metal, composed of electrons and phonons. Using this Hamiltonian a "self-consistent" type of kinetic equation is formulated to describe the response of the electrons *both* to the field and to the phonons. The solution of this equation is accomplished by applying a method recently developed by Oberman[10] for the study of classical systems (see also Dawson and Oberman[11] for similar approach). We thus obtain a closed form for the conductivity due to the electron-phonon interaction, taking into account properly the collective aspects of the electron-electron interaction. We restrict our treatment to intraband transitions only.

II. THE GENERAL FORMALISM

Following Bardeen and Pines,[9] we assume a monoatomic crystal of n ions and n valence electrons per unit volume. We introduce phonon coordinates to represent the ion motion, and second quantization representation

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for the electrons. The phonons are assumed to be either longitudinal or transverse, and only the longitudinal phonons interact with the electrons (long-wavelength region). Creation and annihilation operators, $a_p^\dagger$ and a_p, are defined so that they create (annihilate) an electron in the state $\mathbf{p}$,[12] with the Bloch function ψ_p, and obey the usual anticommutation relations. The Bloch equation is

$$\left[-\frac{1}{2m}\frac{\partial^2}{\partial r^2}+V(\mathbf{r})\right]\psi_p=E(\mathbf{p})\psi_p, \qquad (3)$$

where $V(\mathbf{r})$ stands for the effective potential due to the equilibrium position of the ions compensated by a uniform negative charge. An extended zone scheme is to be used, and the $\mathbf{p}$'s are obtained by the usual boundary conditions. The Hamiltonian for the electrons is

$$H_e=\sum_p E(\mathbf{p})a_p^\dagger a_p+\tfrac{1}{2}\sum_k{}' \phi(k)\rho(\mathbf{k})\rho(-\mathbf{k}), \qquad (4)$$

where

$$\phi(k)=\int d\mathbf{r}d\mathbf{r}'\,\psi_{p+k}(\mathbf{r})\psi_{p'}(\mathbf{r})\frac{e^2}{|\mathbf{r}-\mathbf{r}'|}\psi_{p'+k}(\mathbf{r})\psi_p(\mathbf{r}), \qquad (5)$$

and, the density operator,

$$\rho(\mathbf{k})=\sum_p a^\dagger_{p+k}a_p. \qquad (6)$$

In the spirit of Bloch's theory we assume that $\phi(k)$ depends on the absolute value of the wave vector difference between initial and final states. For free electrons $\phi(k)=4\pi e^2/k^2$.

The longitudinal phonons are represented by the Hamiltonian

$$H_{\text{ph}}=\tfrac{1}{2}\sum_{k(\text{zone})} P_k{}^*P_k+\Omega_k{}^2 Q_k{}^*Q_k, \qquad (7)$$

where Q_k and P_k are the conjugate normal coordinates and momenta, respectively, for phonons in the $\mathbf{k}$ state obeying the usual commutation relations, and $\Omega_k{}^2$ is determined solely by the ion–ion interactions (in the negative background). The summation is restricted to the first Brilliouin zone for the phonons where Q_k and P_k are defined.

The interaction between the electrons and the phonons is represented by the Hamiltonian

$$H_{\text{int}}=\sum_k v_k Q_k\rho(-\mathbf{k}), \qquad (8)$$

where we take

$$v_k=-(nM)^{-1/2}\int d\mathbf{r}\,\psi^*_{p+k}(\mathbf{r})$$
$$\times\left[\sum_j \mathbf{\varepsilon}_k\cdot\frac{\partial}{\partial\mathbf{r}}v(\mathbf{r}-\mathbf{R}_j)e^{i\mathbf{k}\cdot\mathbf{R}_j}\right]\psi_p(\mathbf{r}) \qquad (9)$$

to be the matrix element of the interaction, and to depend only on the wave vector difference between the initial and final states of the electron. We also have $v_k{}^*=v_{-k}$. The sum over $\mathbf{k}$ in Eq. (8) extends over all values, while Q_k refers to the reduced vector in the first zone. This amounts to the inclusion of Peierls-Umklapp processes. In Eq. (9) M is the ion mass, $\mathbf{\varepsilon}_k$ is a unit vector in the $\mathbf{k}$ direction, and $v(\mathbf{r}-\mathbf{R}_j)$ is the effective potential of interaction between an electron in position $\mathbf{r}$ and an ion in equilibrium position $\mathbf{R}_j$. The total Hamiltonian is now

$$H=H_e+H_{\text{ph}}+H_{\text{int}}, \qquad (10)$$

with the irrelevant parts (e.g., the transverse phonons) left out. We also choose our units so that $\hbar=1$.

In order to obtain a self-consistent set of equations, we introduce the density matrix operator, ρ, for a single electron. ρ obeys the equation[13]

$$i\partial\rho/\partial t=[\mathfrak{IC},\rho], \qquad (11)$$

where $\mathfrak{IC}$ is the effective self-consistent Hamiltonian for an electron. In addition to the self-consistent field, there is present a prevailing spatially uniform electric field $\mathbf{E}$, oscillating in time at the frequency ω. For any operator A, we define

$$\langle\mathbf{p}'|A|\mathbf{p}\rangle=\int d\mathbf{r}\,\psi_{p'}{}^*(\mathbf{r})A\psi_p(\mathbf{r}). \qquad (12)$$

The equation of motion for ρ is then given by

$$[i(\partial/\partial t)+E(\mathbf{p})-E(\mathbf{p}+\mathbf{k})]\langle\mathbf{p}+\mathbf{k}|\rho|\mathbf{p}\rangle$$
$$=i(e/m\omega)\mathbf{E}(\omega)\cdot\mathbf{k}\langle\mathbf{p}+\mathbf{k}|\rho|\mathbf{p}\rangle e^{-i\omega t}+\sum_{k'}[\Phi(\mathbf{k}')+v_{k'}Q_{k'}]$$
$$\times[\langle\mathbf{p}+\mathbf{k}-\mathbf{k}'|\rho|\mathbf{p}\rangle-\langle\mathbf{p}+\mathbf{k}|\rho|\mathbf{p}+\mathbf{k}'\rangle], \qquad (13)$$

where

$$\Phi(\mathbf{k})=\phi(k)\sum_p\langle\mathbf{p}+\mathbf{k}|\rho|\mathbf{p}\rangle. \qquad (14)$$

The equation of motion of Q_k is

$$\frac{\partial^2}{\partial t^2}Q_k+\Omega_k{}^2 Q_k=-\mathcal{S}v_k{}^*\sum_p\langle\mathbf{p}+\mathbf{k}|\rho|\mathbf{p}\rangle, \qquad (15)$$

where the $\mathcal{S}$ stands for the sum over all $\mathbf{k}$, which corresponds to this same *reduced* wave vector, so that umklapp processes may be included.

We are concerned here with the solution of Eqs. (13)–(15), under the assumption that the phonons do not respond *directly* to the field. That is to say, the induced current is due only to the electron motion, and is given by

$$\mathbf{j}(\omega)=\mathbf{j}_0(\omega)+\mathbf{j}_1(\omega), \qquad (16)$$

where

$$\mathbf{j}_0(\omega)=i\frac{e^2 n}{m}\frac{1}{\omega}\mathbf{E}(\omega) \qquad (17)$$

[12] We do not introduce spin coordinates because they do not play an explicit role in our theory.

[13] H. Ehrenreich and M. H. Cohen, Phys. Rev. 115, 786 (1959).

is the current for noninteracting electrons, and

$$\mathbf{j}_1(\omega) = -(e/m)\sum_{\mathbf{p}} \mathbf{p}\langle\mathbf{p}|\rho|\mathbf{p}\rangle \tag{18}$$

comes from the solution of Eqs. (13)–(15), neglecting the *direct* effects of the periodicity on the current.

We wish to point out that Eq. (13) is a good approximation for frequencies above the electron–electron collision frequency (which may be a few orders of magnitude smaller than the electron–phonon collision frequency), where one may systematically neglect the electron–electron correlation (including their exchange effects). Thus, our results are only applicable in the limit $\omega\tau > 1$.

III. THE APPROXIMATION METHOD

Adopting Oberman's procedure[10] we solve Eqs. (13)–(15) in four steps.

(A) The equilibrium solution of Eq. (13), *without* taking the phonons and the field into account, is

$$\langle\mathbf{p}+\mathbf{k}|\rho_a|\mathbf{p}\rangle = \delta_{\mathbf{k},0}f_0[E(\mathbf{p})] = f_{\mathbf{p}}, \tag{19}$$

where $f_{\mathbf{p}}$ is the Fermi distribution function

$$f_{\mathbf{p}} = \{\exp\beta[E(\mathbf{p})-\mu]+1\}^{-1}. \tag{20}$$

In Eq. (20), β is the inverse temperature in energy units and μ is the chemical potential of the electrons.

(B) We now incorporate the phonons into Eq. (13) and linearize the equation around the equilibrium solution, Eq. (20), to obtain

$$[i(\partial/\partial t)+E(\mathbf{p})-E(\mathbf{p}+\mathbf{k})]\langle\mathbf{p}+\mathbf{k}|\rho_b|\mathbf{p}\rangle$$
$$= [\Phi_b(\mathbf{k})+v_{\mathbf{k}}Q_{\mathbf{k}}]\{f_{\mathbf{p}}-f_{\mathbf{p}+\mathbf{k}}\}, \tag{21}$$

with

$$\Phi_b(\mathbf{k}) = \phi(k)\sum_{\mathbf{p}}\langle\mathbf{p}+\mathbf{k}|\rho_b|\mathbf{p}\rangle, \tag{22}$$

and

$$\frac{\partial^2}{\partial t^2}Q_{\mathbf{k}}+\Omega_k^2 Q_{\mathbf{k}} = -Sv_{\mathbf{k}}^*\sum_{\mathbf{p}}\langle\mathbf{p}+\mathbf{k}|\rho_b|\mathbf{p}\rangle. \tag{23}$$

Equations (21)–(23) constitute a self-consistent set of equations for the coupled motion of the electrons and lattice waves in the system.

To obtain the steady-state solution of Eqs. (21)–(23), we first perform a Fourier transformation. Then we solve for the electron density

$$n_b(\mathbf{k},\omega) = \sum_{\mathbf{p}}\langle\mathbf{p}+\mathbf{k}|\rho_b|\mathbf{p}\rangle$$

$$= \frac{v_{\mathbf{k}}}{\phi(k)}\frac{1-\epsilon(k,\omega)}{\epsilon(k,\omega)}Q_{\mathbf{k}}(\omega), \tag{24}$$

in terms of $Q_{\mathbf{k}}(\omega)$. In Eq. (24),

$$\epsilon(k,\omega) = 1-\phi(k)\sum_{\mathbf{p}}\frac{f_{\mathbf{p}+\mathbf{k}}-f_{\mathbf{p}}}{E(\mathbf{p}+\mathbf{k})-E(\mathbf{p})-\omega-i\epsilon} \tag{25}$$

is the dielectric function for the electrons. We now substitute Eq. (24) into Eq. (23) and find

$$\left\{-\omega^2+\Omega_k^2+S\frac{|v_k|^2}{\phi(k)}\frac{1-\epsilon(k,\omega)}{\epsilon(k,\omega)}\right\}Q_{\mathbf{k}}(\omega) = 0. \tag{26}$$

Thus, the "real" frequencies ω_k of the $\mathbf{k}$ phonons are given by the equation

$$\omega_k^2 = \Omega_k^2+S\frac{|v_k|^2}{\phi(k)}\frac{1-\epsilon(k,\omega_k)}{\epsilon(k,\omega_k)}. \tag{27}$$

Equation (27) is the well known dispersion relation for the phonons, and was derived by Nakajima (see Ref. 9, Bardeen-Pines) by a method of canonical transformation. Noticing that $\omega_k \lesssim 10^{-4}E_F$, where E_F is the Fermi energy, Eq. (27) is reduced to

$$\omega_k^2 = \Omega_k^2+\frac{|v_k|^2}{\phi(k)}\frac{1-\epsilon(k,0)}{\epsilon(k,0)}, \tag{28}$$

similar to the result of Bardeen.[14] In the plasma model for the metal, the dielectric function is given by

$$\epsilon(k,0) \approx 1+K^2/k^2, \tag{29}$$

where

$$K^2 = (4m/\pi h)(v_F/a) \tag{30}$$

is the square of the inverse shielding radius, $a = h^2/me^2$ is the Bohr radius, and v_F is Fermi velocity. Also

$$\Omega_k^2 \approx \Omega_p^2 = 4\pi e^2 n/M \tag{31}$$

and

$$v_k \approx -i(4\pi e^2/k)(n/M)^{1/2} \tag{32}$$

and, thus,

$$\omega_k = ck, \tag{33}$$

with c, the sound velocity, given by

$$c^2 = \Omega_p^2/K^2 = (m/3M)v_F^2. \tag{33a}$$

This result was obtained first by Bohm and Staver.[15]

In the preceding paragraph we were able to derive the spectrum of the phonons, but the amplitudes of the $Q_{\mathbf{k}}(\omega_k)$ are not defined by Eqs. (21)–(23). However, for the time being, we shall assume $Q_{\mathbf{k}}(\omega_k)$ to be given. It is now convenient to introduce the creation and annihilation operators for the phonons by the equation

$$Q_{\mathbf{k}} = (2\omega_k)^{-1/2}[b_k+b_{-k}^\dagger], \tag{34}$$

and, thus, the time dependence of $Q_{\mathbf{k}}$ is readily given by

$$Q_{\mathbf{k}}(t) = (2\omega_k)^{-1/2}\{b_k e^{-i\omega_k t}+b_{-k}^\dagger e^{i\omega_k t}\}, \tag{35}$$

where ω_k is the "positive" solution of Eq. (27).

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[15] D. Bohm and T. Staver, Phys. Rev. **84**, 836 (1952) and T. Staver Ph.D. thesis, Princeton University, 1952 (unpublished).

Going back to Eq. (21) we obtain

$$n_b(\mathbf{k},t)=\frac{v_k}{(2\omega_k)^{1/2}\phi(k)}\left\{\frac{1-\epsilon(k,\omega_k)}{\epsilon(k,\omega_k)}b_k e^{-i\omega_k t}\right.$$

$$\left.+\frac{1-\epsilon^*(k,\omega_k)}{\epsilon^*(k,\omega_k)}b_{-k}^\dagger e^{i\omega_k t}\right\}, \quad (36)$$

where use has been made of the property

$$\epsilon(k,-\omega)=\epsilon^*(k,\omega). \quad (37)$$

Consequently, the time-dependent $\langle\mathbf{p}+\mathbf{k}|\rho_b|\mathbf{p}\rangle$ reads

$$\langle\mathbf{p}+\mathbf{k}|\rho_b(t)|\mathbf{p}\rangle$$

$$=\frac{v_k}{(2\omega_k)^{1/2}}\left[\frac{1}{\epsilon(k,\omega_k)}\frac{f_{p+k}-f_p}{E(\mathbf{p}+\mathbf{k})-E(\mathbf{p})-\omega_k-i\epsilon}b_k e^{-i\omega_k t}\right.$$

$$\left.+\frac{1}{\epsilon^*(k,\omega_k)}\frac{f_{p+k}-f_p}{E(\mathbf{p}+\mathbf{k})-E(\mathbf{p})+\omega_k-i\epsilon}b_{-k}^\dagger e^{i\omega_k t}\right]. \quad (38)$$

(C) In the third step we find the response of the electrons to the local prevailing $\mathbf{E}$ field. This is accomplished by rewriting Eq. (13) as

$$[i(\partial/\partial t)+E(\mathbf{p})-E(\mathbf{p}+\mathbf{k})]\langle\mathbf{p}+\mathbf{k}|\rho_c|\mathbf{p}\rangle$$
$$+\Phi_c(\mathbf{k})[f_{p+k}-f_p]$$
$$=i(e/m\omega)\mathbf{E}\cdot\mathbf{k}\langle\mathbf{p}+\mathbf{k}|\rho_b(t)|\mathbf{p}\rangle e^{-i\omega t}, \quad (39)$$

and

$$\Phi_c(\mathbf{k})=\phi(k)\sum_p\langle\mathbf{p}+\mathbf{k}|\rho_c|\mathbf{p}\rangle=\phi(k)n_c(\mathbf{k}). \quad (40)$$

Noticing that Eq. (36) is driven by the frequencies $\omega+\omega_k$ and $\omega-\omega_k$, we obtain

$$n_c(\mathbf{k},t)=\sum_p\langle\mathbf{p}+\mathbf{k}|\rho_c(t)|\mathbf{p}\rangle=i\frac{e}{m\omega^2}\mathbf{k}\cdot\mathbf{E}(\omega)\frac{v_k}{\phi(k)}\frac{1}{(2\omega_k)^{1/2}}$$

$$\times\left\{b_k\left[\frac{1}{\epsilon(k,\omega_k)}-\frac{1}{\epsilon(k,\omega+\omega_k)}\right]e^{-i(\omega+\omega_k)t}\right.$$

$$\left.+b_{-k}^\dagger\left[\frac{1}{\epsilon^*(k,\omega_k)}-\frac{1}{\epsilon(k,\omega-\omega_k)}\right]e^{-i(\omega-\omega_k)t}\right\}, \quad (41)$$

where $n_c(k,t)$ is the response of the electron density to both the field $\mathbf{E}$ and the phonons motion.

(D) The last step is the current calculation. To compute the average current induced by the field we take Eq. (13) for $\mathbf{k}=0$, multiply it by $-(e/m)\mathbf{p}$, and sum over all $\mathbf{p}$. If we neglect all the *direct* effects of the periodicity, but still consider the electron-phonon umklapp processes, we obtain

$$i\frac{\partial}{\partial t}\mathbf{j}_1(t)=\sum_k[\Phi(\mathbf{k},t)+v_k Q_k(t)]$$

$$\times\sum_p\left(-\frac{e}{m}\mathbf{p}\right)\{\langle\mathbf{p}-\mathbf{k}'|\rho|\mathbf{p}\rangle-\langle\mathbf{p}|\rho|\mathbf{p}+\mathbf{k}'\rangle\}$$

$$=\frac{e}{m}\sum_k[\Phi(-\mathbf{k},t)+v_k^*Q_k^*(t)]\mathbf{k}$$

$$\times\sum_p\langle\mathbf{p}+\mathbf{k}|\rho|\mathbf{p}\rangle. \quad (42)$$

Making use of the fact that $\Phi(-\mathbf{k})=\Phi(\mathbf{k})$, Eq. (42) reduces to

$$i\frac{\partial}{\partial t}\mathbf{j}_1(t)=\frac{e}{m}\sum_k\mathbf{k}\frac{v_k^*}{(2\omega_k)^{1/2}}[b_k^\dagger e^{i\omega_k t}+b_{-k}e^{-i\omega_k t}]$$

$$\times\sum_p\langle\mathbf{p}+\mathbf{k}|\rho|\mathbf{p}\rangle. \quad (43)$$

The coherent part of the current induced by the $\mathbf{E}$ field is given by

$$\omega_1\mathbf{j}(\omega)e^{-i\omega t}=\frac{e}{m}\sum_k\mathbf{k}\frac{v_k^*}{(2\omega_k)^{1/2}}$$

$$\times[b_k^\dagger e^{i\omega_k t}+b_{-k}e^{-i\omega_k t}]n_c(\mathbf{k},t), \quad (44)$$

where $n_c(\mathbf{k},t)$ comes from Eq. (41), and, thus,

$$\mathbf{j}_1(\omega)=i\frac{e^2}{m^2\omega^3}\sum_k\mathbf{kk}\cdot\mathbf{E}\frac{|v_k|^2}{2\omega_k\phi(k)}\left\{\left[\frac{1}{\epsilon(k,\omega_k)}+\frac{1}{\epsilon^*(k,\omega_k)}\right]\right.$$

$$\left.-\left[\frac{1}{\epsilon(k,\omega+\omega_k)}+\frac{1}{\epsilon(k,\omega-\omega_k)}\right]\right\}(b_k^\dagger b_k+\tfrac{1}{2}). \quad (45)$$

In the derivation of Eq. (45) we have used the properly *symmetrized* form of Eq. (44) and the commutation relations of the b_k's.

IV. THE CONDUCTIVITY AND THE RELAXATION TIME

So far we have calculated the response of the system to the $\mathbf{E}$ field and to the phonons motion. However, for the calculation of the frequency-dependent conductivity we have to specify the phonon spectrum. Here we shall assume, following Bloch, that the phonon system may be treated as if it were in *thermal equilibrium*. Thus, the ensemble average of the current of Eq. (45) is given in terms of

$$\langle b_k^\dagger b_k\rangle=\{e^{\beta\omega_k}-1\}^{-1}, \quad (46)$$

and, consequently,

$$\langle b_k^\dagger b_k\rangle+\tfrac{1}{2}=\tfrac{1}{2}\coth(\beta\omega_k/2). \quad (47)$$

The conductivity, $\sigma(\omega)$, is defined by the equation

$$\mathbf{j}(\omega)=\sigma(\omega)\mathbf{E}(\omega). \quad (48)$$

If we employ Eqs. (18) and (45) we can write

$$\sigma(\omega)=\sigma_0(\omega)[1-(2e^2/3\pi m\omega^2)F(\omega)], \quad (49)$$

where

$$\sigma_0=i(e^2 n/m\omega) \quad (50)$$

is the conductivity for free electrons, and

$$F(\omega) = \sum_{|k|} k^4 \frac{|v_k|^2}{e^2 n \phi(k)} \left\{ \left[\frac{1}{\epsilon(k,\omega_k)} + \frac{1}{\epsilon^*(k,\omega_k)} \right] \right.$$
$$\left. - \left[\frac{1}{\epsilon(k,\omega+\omega_k)} + \frac{1}{\epsilon(k,\omega-\omega_k)} \right] \right\} \frac{\coth(\beta\omega_k/2)}{\omega_k} . \quad (51)$$

In the derivation of Eq. (51) we have carried out a summation over all the directions of $\mathbf{k}$.

In order to cast Eq. (49) in the form of Eq. (2), we notice that the second term in the brackets of Eq. (49) is much *smaller* than one, and thus we can write

$$\sigma(\omega) \approx i \frac{e^2 n}{m^*} \frac{1}{\omega + i/\tau(\omega)} , \quad (52)$$

where

$$m^* \approx m\{1 + (2e^2/3\pi m\omega^2) \operatorname{Re}[F(\omega)]\} \quad (53)$$

is the "renormalized mass" of the electrons, and

$$[\tau(\omega)]^{-1} = + (2e^2/3\pi m\omega) \operatorname{Im}[F(\omega)] \quad (54)$$

is the frequency-dependent relaxation time. In Eqs. (53) and (54) Re and Im stand for the real and imaginary parts, respectively. We wish to point out that the "renormalization" of the mass is due to the phonon field and *not* due to the usual lattice effects. [These should be taken care of by the mass, m, on the right-hand side of Eq. (53).]

An interesting aspect of Eq. (54) is that even for *zero temperature* the imaginary part of $F(\omega)$ [see Eq. (51)] does not vanish, and the resistivity is finite at high frequencies. This is due to the collision of the *excited* electrons with the phonon fluctuations at zero temperature. A similar effect is well known in the theory of x-ray diffraction in nonconducting crystals (see Peierls[2]).

To conclude, we make some further simplification of Eq. (51), assuming the plasma model for a metal. First, we ignore the ω_k in the dielectric functions, and replace the summation over k by integration. Thus, Eq. (51) reads

$$F(\omega) = \frac{2}{2\pi} \int dk \, k^4 \frac{|v_k|^2}{e^2 n \phi(k)}$$
$$\times \left\{ \frac{1}{\epsilon(k,0)} - \frac{1}{\epsilon(k,\omega)} \right\} \frac{\coth(\beta\omega_k/2)}{\omega_k} . \quad (55)$$

Next we use Eqs. (32) and (33) with $\phi(k) = 4\pi e^2/k^2$ and obtain

$$F(\omega) = \frac{1}{\pi} \frac{1}{Mc} \int dk \, k^3 \left\{ \frac{1}{\epsilon(k,0)} - \frac{1}{\epsilon(k,\omega)} \right\}$$
$$\times \coth(\beta ck/2), \quad (56)$$

where c is the sound velocity. Finally, we write for zero temperature

$$F(\omega) = \frac{1}{\pi} \frac{1}{Mc} \int dk \, k^3 \left\{ \frac{1}{\epsilon(k,0)} - \frac{1}{\epsilon(k,\omega)} \right\} , \quad (57)$$

and for temperatures above Debye temperature, θ,

$$F(\omega) = \frac{1}{\pi} \frac{2}{\beta M c^2} \int dk \, k^2 \left\{ \frac{1}{\epsilon(k,0)} - \frac{1}{\epsilon(k,\omega)} \right\} ,$$
$$(1/\beta \gg \theta). \quad (58)$$

Let us examine some of the features of these results, which we presume are valid in general. At high frequencies and low temperatures the relaxation time does not approach zero like T^5, as in the low-frequency case, but rather approaches a finite value given by Eq. (57). For high temperatures one finds the usual T behavior, as may be expected on the basis of the well-known arguments of the low-frequency case. The dependence on the ion mass, M, is different at high and low temperatures. While for low temperatures $F(\omega)$ is proportional to $M^{-1/2}$ [compare with Migdal[16]], in the high-temperature case it is independent of the ion mass.

We have calculated in the present paper only the contribution of the electron-photon interaction to the relaxation time, but the method discussed above may be as well applied to the impurities contribution in metals.

ACKNOWLEDGMENTS

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